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ON SOME GROUPS RELATED TO ARC-COLORED DIGRAPHS^(*)

An arc-coloring of a digraph G is an assignation of colors to its arcs in such a way that all arcs incident to each vertex, as well as all arcs incident from each vertex, have different colors. In an arc-colored d -regular digraph, each color can be associated to a permutation of the vertices of the digraph. In this paper we show the relation between the permutation group generated by these permutations and the groups of those automorphisms of the arc-colored digraph which preserve or exchange colors.

1 Introduction

An arc-coloring of a digraph G is an assignation of colors to its arcs in such a way that all arcs incident to each vertex, as well as all arcs incident from each vertex, have different colors. The automorphisms of the digraph which preserve colors or exchange them are called s -colored and colored automorphisms respectively. These definitions are made precise in Section 2. When the digraph is d -regular (and d colors are used), each color can be associated to a permutation on the vertex set of G . It was shown in [3] that, in the case of Cayley digraphs, there is a relation between the permutation group generated by the permutations associated to each color and the groups of colored and s -colored automorphisms of the digraph. In Section 3, a similar result for strongly connected Schreier digraphs is obtained. In the next section, we show that any strongly connected d -regular digraph G is isomorphic to an Schreier digraph, so that, for each

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arc-coloring of G , we can obtain its colored automorphism group in terms of the permutation group of the arc-coloring.

2 Basic Definitions and Terminology

Throughout this paper, G denotes a d -regular and strongly connected directed graph (digraph), with set of vertices $V = V(G)$ and set of arcs $A = A(G)$. Basic terminology from graph theory can be found for instance in [1]. We allow loops and multiple arcs.

An arc-coloring of G with set of "colors" $C = \{1, 2, \dots, d\}$ is a map $\chi : A \rightarrow C$ such that, for each vertex $x \in V$, χ assigns different colors to the arcs incident to x as well as to the arcs incident from x . That is, the maps $\chi[\cdot, x] : \Gamma^-(x) \rightarrow C$ and $\chi[x, \cdot] : \Gamma^+(x) \rightarrow C$ are bijective $\forall x \in V$. Any arc-coloring χ of G induces a set $\Delta = \{\delta_1, \dots, \delta_d\}$ of permutations of the vertices of G defined by

$$(1) \quad \delta_i(x) = y \Leftrightarrow [x, y] \in A \quad \text{and} \quad \chi[x, y] = i$$

Reciprocally, let $\Delta = \{\delta_1, \dots, \delta_d\}$ be a set of d permutations of V such that

- (i) $\delta_i(x) \in \Gamma^+(x)$, $i = 1, \dots, d$, $\forall x \in V$, and
- (ii) $\delta_i(x) \neq \delta_j(x)$ whenever $i \neq j$, $\forall x \in V$.

Then Δ induces univocally an arc-coloring of G defined as $\chi[x, y] = i$ iff $y = \delta_i(x)$. The set Δ is said to be a *decomposition into permutations* of G , see [2]. The two definitions are equivalent and will be indistinctly used. In what follows, we identify through (1) the set of colors C with the set of permutations Δ . The arc-colored digraph (G, Δ) is the digraph G together with the arc-coloring associated to Δ . It is not difficult to prove that any regular digraph admits (usually several) arc-colorings. A proof can be found for instance in [2]. For simplicity, the prefix arc- will be omitted in our terminology, although it must be understood.

Two colored digraphs (G, Δ) and (G', Δ') are said to be *isomorphic* if there exist a digraph isomorphism $\phi : G \rightarrow G'$ and a bijection $\sigma : \Delta \rightarrow \Delta'$ such that $\phi(\delta_i(x)) = \sigma(\delta_i)\phi(x)$, $i = 1, \dots, d$, $\forall x \in V$. We also say that ϕ is a *colored isomorphism* between (G, Δ) and (G', Δ') .

Similarly, an automorphism φ of the digraph G is said to be a *colored automorphism* of (G, Δ) if there exists a permutation σ on $\{1, 2, \dots, d\}$ such that $\varphi(\delta_i(x)) = \delta_{\sigma(i)}\varphi(x)$, $i = 1, \dots, d$, $\forall x \in V$. When σ is the identity, φ is said to be an *strictly colored*, or simply *s-colored*, automorphism of (G, Δ) . We consider three groups associated to a colored digraph (G, Δ) . The *permutation group* of (G, Δ) is the subgroup of $\text{Sym } V$ generated by

the permutations in Δ , and it is denoted by $\Sigma = \Sigma(G, \Delta)$. The *colored* group of (G, Δ) , $\text{Aut}(G, \Delta)$, is the group of its colored automorphisms, and the *s-colored* group of (G, Δ) , $s\text{-Aut}(G, \Delta)$, is the group of its *s-colored* automorphisms. It is easily checked that $s\text{-Aut}(G, \Delta)$ is a normal subgroup of $\text{Aut}(G, \Delta)$.

Finally, let us review some group theoretic concepts. They can be found for instance in [5]. A permutation group Σ acting on a set V is said to be *transitive* if, for any $x, y \in V$, there exists γ in Σ such that $\gamma(x) = y$. In our context, $\Sigma(G, \Delta)$ is transitive iff G is strongly connected, that is, there exists a path from each vertex to any other. The stabilizer Σ_X of a subset $X \subset V$ is the subgroup $\Sigma_X = \{\gamma \in \Sigma : \gamma(X) = X\}$. When $X = \{x\}$, Σ_X is called the stabilizer of the element x , and we write Σ_x for $\Sigma_{\{x\}}$. If Σ is transitive, the stabilizers of any two elements in V are conjugate subgroups in Σ and $|\Sigma| = |V||\Sigma_x|$. When $|\Sigma_x| = 1 \forall x \in V$, Σ is said to be *semiregular*, and *regular* if Σ is also transitive. We write the composition of permutations in the usual way, that is, $\sigma \circ \sigma'(x) = \sigma(\sigma'(x))$, and the identity permutation is denoted by ι .

Given two groups Ω and H and a group homomorphism $\psi: H \rightarrow \text{Aut}\Omega$, the *semidirect product* $\Omega \rtimes_\psi H$ is the group $(\Omega \times H, *)$ where the composition law is defined by

$$(\omega, h) * (\omega', h') = (\omega\psi(h)(\omega'), hh')$$

Then, $\Omega \times \{\iota\}$ and $\{e\} \times H$ are respectively a normal subgroup and a subgroup of $\Omega \rtimes_\psi H$, which will also be denoted by Ω and H . Moreover, $\Omega \cap H$ is the identity and ΩH is the full semidirect product.

3 Cayley and Schreier digraphs and their groups

Cayley digraphs are the most simple examples of colored digraphs. Given a group Ω with identity element e , and a set $\Delta = \{a_1, \dots, a_d\}$ of generators of Ω , the (left-) Cayley digraph of Ω with respect to Δ is the colored d -regular digraph $\text{Cay}(\Omega, \Delta)$, or simply (Ω, Δ) , with

$$\begin{cases} V(\Omega, \Delta) = \Omega, \\ A(\Omega, \Delta) = \{[\omega, a_i\omega]; \omega \in \Omega, a_i \in \Delta\}, \text{ and} \\ \chi[\omega, a_i\omega] = i \quad \text{or} \quad \delta_i(\omega) = a_i\omega. \end{cases}$$

The permutation group $\Sigma(\Omega, \Delta)$ is just the left regular representation of Ω , so that $\Sigma(\Omega, \Delta) \cong \Omega$. It is easily checked that right translations in Ω are *s-colored* automorphisms of the digraph (Ω, Δ) . Indeed, it can be seen

that the map $\Psi: \Omega \rightarrow s\text{-Aut}(\Omega, \Delta)$, where, $\Psi(\omega)(x) = x\omega^{-1}$, $x \in \Omega$, is a group isomorphism, see for instance [5]. In particular, this result shows that a Cayley digraph is vertex transitive. On the other hand, let H be the subgroup of those automorphisms of the group Ω which leave Δ invariant, that is, $H = \{\sigma \in \text{Aut } \Omega : \sigma(\Delta) = \Delta\}$. Then, it was proved in [3] that the map $\Phi: \Omega \times_{\iota} H \rightarrow \text{Aut}(\Omega, \Delta)$ defined as $\Phi(\omega, \pi)(x) = \pi(x)\omega^{-1}$, $x \in \Omega$, is a group isomorphism (ι denotes the canonical injection of H in $\text{Aut } \Omega$). These results can be summarized in the following theorem.

THEOREM (3.1) — *Let (Ω, Δ) be a Cayley digraph of Ω with respect to the set Δ of generators. Then,*

- (i) $\Sigma(\Omega, \Delta) \cong \Omega$.
- (ii) $s\text{-Aut}(\Omega, \Delta) \cong \Omega$.
- (iii) $\text{Aut}(\Omega, \Delta) \cong \Omega \times_{\iota} H$.

□

It is not difficult to see that $s\text{-Aut}(G, \Delta)$ is always semiregular in its action over V , while $\Sigma(G, \Delta)$ is always transitive (in a strongly connected digraph G). Therefore, the Cayley digraphs have the largest s -colored group and the smallest permutation group. On the other hand, it can also be easily seen that any colored digraph is a Cayley digraph iff this property holds. In other words, (G, Δ) is a Cayley digraph iff $s\text{-Aut}(G, \Delta) \cong \Sigma(G, \Delta)$.

Theorem 3.1 is also true when we consider right-Cayley digraphs, in which the arcs are defined as $\omega \rightarrow \omega a_i$. In this case, the right representation of Ω is $v: \Omega \rightarrow \Sigma$, $v(\omega)x = x\omega^{-1}$, $x \in \Omega$, the isomorphism $\Psi: \Omega \rightarrow s\text{-Aut}(\Omega, \Delta)$ is $\Psi(\omega)(x) = \omega x$, $x \in \Omega$, and the isomorphism $\Phi: \Omega \times_{\iota} H \rightarrow \text{Aut}(\Omega, \Delta)$ is $\Phi(\omega, \sigma)(x) = \omega\sigma(x)$.

Now, let S be a subgroup of Ω . The (left-)Schreier diagram of Ω modulo S with respect to Δ is the colored d -regular digraph $\text{Sch}(\Omega/S, \Delta)$, or simply $(\Omega/S, \Delta)$, with

$$\begin{cases} V(\Omega/S, \Delta) = \{\omega S; \omega \in \Omega\} \\ A(\Omega/S, \Delta) = \{[\omega S, a_i \omega S], \omega \in \Omega, a_i \in \Delta\} \\ \chi[\omega S, a_i \omega S] \quad \text{or} \quad \delta_i(\omega S) = a_i \omega S; \end{cases}$$

We will always assume that no two elements in Δ belong to the same right coset modulo S . The map $v: \Omega \rightarrow \Sigma(\Omega/S, \Delta)$ given by $v(\omega)(xS) = \omega xS$, $x \in \Omega$, is the representation of Ω on the set of left cosets modulo S ; hence, $\Sigma = \Sigma(\Omega/S, \Delta) \cong \Omega/S_{\Omega}$, where $S_{\Omega} = \cap_{\omega \in \Omega} \omega^{-1}S\omega$ is the so-called core of S in Ω . We denote $v(\omega)$ by $\gamma_{\omega} \in \Sigma$. Of course, when S is the trivial group, we have the Cayley digraph of Ω with respect to Δ , so that Schreier digraphs are a generalization of Cayley digraphs. This fact suggests that a result similar to Theorem 3.1 can be obtained for Schreier digraphs. Indeed, we

show that all colored automorphisms of $(\Omega/S, \Delta)$ are induced by the colored automorphisms of (Ω, Δ) which are compatible with the relation modulo S .

Let φ be a colored automorphism of $(\Omega/S, \Delta)$ and σ its associated permutation on $\{1, \dots, d\}$, $\varphi(\delta_i(\omega S)) = \delta_{\sigma(i)}\varphi(\omega S)$, $\delta_i \in \Delta$, $\omega \in \Omega$. Let $\gamma = \delta_{i_s} \circ \dots \circ \delta_{i_1} = \delta_{j_r} \circ \dots \circ \delta_{j_1}$ be two decompositions of $\gamma \in \Sigma$ in terms of the elements in Δ . Then, $\varphi(\gamma(\omega S)) = \delta_{\sigma(i_s)} \circ \dots \circ \delta_{\sigma(i_1)}\varphi(\omega S) = \delta_{\sigma(j_r)} \circ \dots \circ \delta_{\sigma(j_1)}\varphi(\omega S) \forall \omega \in \Omega$, so that $\delta_{\sigma(i_s)} \circ \dots \circ \delta_{\sigma(i_1)} = \delta_{\sigma(j_r)} \circ \dots \circ \delta_{\sigma(j_1)}$. This fact tells us that the map $\bar{\sigma}: \Sigma \rightarrow \Sigma$ given by $\bar{\sigma}(\gamma) = \bar{\sigma}(\delta_{i_s} \circ \dots \circ \delta_{i_1}) = \delta_{\sigma(i_s)} \circ \dots \circ \delta_{\sigma(i_1)}$ is well defined. Clearly, $\bar{\sigma}$ is an automorphism of the group Σ which we denote also by σ . If $\varphi(S) = \omega_\varphi S$, we can write

$$\varphi(\omega S) = \varphi(\gamma_\omega(S)) = \sigma(\gamma_\omega)(\omega_\varphi S), \forall \omega \in \Omega$$

so that φ is determined by σ and the image of a single element in Ω/S . Moreover, the stabilizer in Σ of S is $\Sigma_S = \nu(S) = \{\gamma_\omega : \omega \in S\}$, and if $\omega \in S$, then $\sigma(\gamma_\omega)(\omega_\varphi S) = \varphi(\omega S) = \varphi(S) = \omega_\varphi S$, hence $\sigma(\Sigma_S)$ is the stabilizer in Σ of $\omega_\varphi S$, $\sigma(\Sigma_S) = \gamma_{\omega_\varphi} \Sigma_S \gamma_{\omega_\varphi}^{-1}$. These remarks lead to the following result. As in the case of Cayley digraphs, $H = \{\sigma \in \text{Aut } \Omega : \sigma(\Delta) = \Delta\}$.

THEOREM (3.2) — Let $B = \{(\gamma, \pi) \in \Sigma \times_\iota H : \pi(\Sigma_S) = \gamma^{-1} \Sigma_S \gamma \text{ and } N_\Sigma(\Sigma_S) \text{ the normalizer of } \Sigma_S \text{ in } \Sigma\}$. Then

- (i) $\text{Aut}(\Omega/S, \Delta) \cong B/\Sigma_S$.
- (ii) $s - \text{Aut}(\Omega/S, \Delta) \cong N_\Sigma(\Sigma_S)/\Sigma_S$.

PROOF — Let $\Phi: B \rightarrow \text{Aut}(\Omega/S, \Delta)$ be defined as $\Phi(\gamma, \sigma)(\omega S) = \Phi(\gamma, \sigma)(\gamma_\omega(S)) = \sigma(\gamma_\omega)\gamma^{-1}(S)$. If $\omega S = \omega' S$, $\gamma_{\omega^{-1}\omega'} \in \Sigma_S$ and

$$(\sigma(\gamma_\omega)\gamma^{-1})^{-1}(\sigma(\gamma_{\omega'})\gamma^{-1})(S) = \gamma\sigma(\gamma_{\omega^{-1}\omega'})\gamma^{-1}(S) = S,$$

hence $\sigma(\gamma_\omega)\gamma^{-1}(S) = \sigma(\gamma_{\omega'})\gamma^{-1}(S)$ and $\Phi(\gamma, \sigma)$ is well defined. Clearly,

$$\Phi(\gamma, \sigma)(a_i \omega S) = \sigma(\gamma_{a_i \omega})\gamma^{-1}(S) = \sigma(\gamma_{a_i})\sigma(\gamma_\omega)\gamma^{-1}(S) = \delta_{\sigma(i)}\Phi(\gamma, \sigma)(\omega S)$$

so that $\Phi(\gamma, \sigma) \in \text{Aut}(\Omega/S, \Delta)$. Moreover,

$$\begin{aligned} \Phi(\gamma, \sigma) \circ \Phi(\gamma', \sigma')(\omega S) &= \Phi(\gamma, \sigma)(\sigma'(\gamma_\omega)\gamma'^{-1}(S)) = \sigma(\sigma'(\gamma_\omega)\gamma'^{-1})\gamma^{-1}(S) = \\ &= \sigma\sigma'(\gamma_\omega)(\gamma\sigma(\gamma'))^{-1}(S) = \Phi(\gamma\sigma(\gamma'), \sigma\sigma')(\omega S), \end{aligned}$$

so that Φ is a group homomorphism. From the remarks above, we know that Φ is also exhaustive. Finally, $\Phi(\gamma, \sigma) = \iota$ implies $\Phi(\gamma, \sigma)(S) = \gamma^{-1}(S) = S$, hence $\gamma \in \Sigma_S$, and then, $\sigma = \iota$. Therefore, $\text{Ker } \Phi = \Sigma_S$. This proves (i). Now, φ is an s -colored automorphism iff $\sigma = \iota$. But $(\gamma, \iota) \in B$ iff

$\gamma \in N_\Sigma(\Sigma_S)$. This proves (ii). \square

When the core of S in Ω , S_Ω , is trivial, the above results become

- (i) $\Sigma(\Omega/S, \Delta) \cong \Omega$
- (ii) $\text{Aut}(\Omega, \Delta) \cong \{(\gamma, \sigma) \in \Omega \times_\iota H : \sigma(S) = \gamma S \gamma^{-1}\} / S$,
- (iii) $s\text{-Aut}(\Omega, \Delta) \cong N_\Omega(S)$

where the comparison with Theorem 3.1 is made in a more clear way.

As in the case of Cayley digraphs, we can restate the above result in the case of right Schreier digraphs with minor modifications in the isomorphisms involved. Finally, we would remark that, although not strictly necessary, there is no loss of generality in assuming that Δ is a set of generators of Ω . In fact, if $\langle \Delta \rangle = \Omega' < \Omega$ and $(\Omega/S, \Delta)$ is strongly connected, the Schreier digraphs $(\Omega/S, \Delta)$ and $(\Omega'/S', \Delta)$, where $S' = S \cap \Omega'$ are isomorphic in the obvious way.

4 The groups of an arc-colored digraph

The importance of Schreier digraphs arises from the fact that any strongly connected d -regular digraph is isomorphic to an Schreier digraph. More precisely, let G be a strongly connected d -regular digraph, Δ an (arc) coloring of G and x a distinguished vertex in G . As usual, $\Sigma = \Sigma(G, \Delta)$ denotes the permutation group associated to the coloring Δ , and Σ_x the stabilizer of x in Σ .

PROPOSITION (4.1) — *The colored digraph (G, Δ) is isomorphic to the Schreier digraph $(\Sigma/\Sigma_x, \Delta)$.*

PROOF — For any $y \in V$, denote by γ_y any element in $\Sigma = \Sigma(G, \Delta)$ such that $\gamma_y(x) = y$. Consider the map $\psi: (G, \Delta) \rightarrow (\Sigma/\Sigma_x, \Delta)$ defined as $\psi(y) = \psi(\gamma_y(x)) = \gamma_y \Sigma_x$. It is easily checked that ψ is well defined and bijective. Moreover, $\psi(\delta_i(y)) = \psi(\delta_i \gamma_y(x)) = \delta_i \gamma_y \Sigma_x = \delta_i \psi(y) \forall y \in V, \forall \delta_i \in \Delta$. Hence, ψ is a colored isomorphism between the two digraphs. \square

This last proof is essentially the same as that used in [4] to prove that any regular graph of even degree is isomorphic to an Schreier graph. There, an Schreier graph was defined as the undirected version of an Schreier digraph.

Notice that if Σ_x contains a normal subgroup N of Σ , we must have $N < \bigcap_{y \in V} \Sigma_y = \{i\}$. In other words, the core of Σ_x in Σ is trivial.

Therefore, Theorem 3.2 can be restated in the following way. As before, $H = \{\sigma \in \text{Aut}(\Sigma) : \sigma(\Delta) = \Delta\}$ and $B = \{(\gamma, \sigma) \in \Sigma \times H : \sigma(\Sigma_x) = \gamma^{-1} \Sigma_x \gamma\}$.

THEOREM (4.2) - Let (G, Δ) be a colored digraph, $x \in V(G)$ and $\Sigma = \Sigma(G, \Delta)$.

Then,

(i) $Aut(G, \Delta) \cong B/\Sigma_x$.

(ii) $s\text{-}Aut(G, \Delta) \cong N_\Sigma(\Sigma_x)/\Sigma_x$. □

Since $\gamma\Sigma_x\gamma^{-1} = \Sigma_{\gamma(x)}$, we have $\gamma \in N_\Sigma(\Sigma_x)$ iff $\Sigma_x = \Sigma_{\gamma(x)}$. In other words, the order of $s\text{-}Aut(G, \Delta)$ is the order of the largest fixed block of Σ_x . For any element y left fixed by Σ_x , there exists an s -colored automorphism φ which sends x to y , namely, $\Phi(\gamma_y^{-1}, \iota)$, where Φ is the isomorphism defined in the proof of Theorem 3.2. According to the definition of Φ , $\varphi(z) = \varphi(\gamma_z(x)) = \gamma_z \circ \gamma_y(x) = \gamma_z(y)$.

Notice that if $\varphi \in AutG$ and χ is a coloring of G , φ induces a coloring χ_φ in G defined as $\chi_\varphi[x, y] = \chi[\varphi(x), \varphi(y)]$. Hence, when G is uniquely arc-colorable (up to colored isomorphisms), then $\varphi \in Aut(G, \Delta)$, or $AutG = Aut(G, \Delta)$.

As a final remark, we would mention that any graph can be seen as a symmetric digraph. Hence, the results here obtained can be applied to the class of d -regular graphs which are d -edge-colorable.

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